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Netlike partial cubes, IV: Fixed finite subgraph theorems

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ABSTRACT

We prove that, if a netlike partial cube G (see [N. Polat, Netlike partial cubes I. General properties, Discrete Math. 307 (2007) 2704–2722]) contains no isometric rays, then there exists a convex cycle or a finite hypercube which is fixed by every automorphism of G. Furthermore we prove that every self-contraction (map which preserves or collapses the edges) of G fixes a convex cycle or a finite hypercube if and only if G contains no isometric rays. We also study the self-contractions of G which fix no finite set of vertices.

1. Introduction

The class of netlike partial cubes was introduced in Part I [16] of this series of papers as a class of partial cubes containing median graphs, even cycles, benzenoid graphs and cellular bipartite graphs as particular elements.

In this fourth paper we pursue the study of netlike partial cubes by focusing on fixed subgraph properties, and chiefly by generalizing three results of Tardif [18] on median graphs. Fixed subgraph theorems are far-reaching outgrowths of metric fixed point theory. They have been a flourishing topic in the recent literature on metric graph theory. See in particular the study [4] by Brešar et al. of tree-like partial cubes, another class of finite partial cubes that contains all finite median graphs.

For a netlike partial cube G, just as for median graphs, the property that every self-contraction fixes a finite regular netlike subgraph is directly linked to the absence of isometric rays in G. The proofs of this result and of related ones, which form the best part of this paper, require the geodesic topology, a topology which was introduced in [11] for the study of graphs containing no isometric rays, and which turns out for netlike partial cubes to be the topology generated by the convex sets as a subbase.

In the last section we use this topology to specify which ends of a netlike partial cube are directions of translating self-contractions of this graph, namely self-contractions which fix no finite set of vertices.

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2. Preliminaries

2.1. Graphs

The graphs we consider are undirected, without loops or multiple edges, and may be finite or infinite. Let G be a graph. If $x \in V(G)$, the set $N_G(x) := \{y \in V(G) : xy \in E(G)\}$ is the *neighborhood* of x in G, $N_G[x] := \{x\} \cup N_G(x)$ is the *closed neighborhood* of x in G and $\delta_G(x) := |N_G(x)|$ is the *degree* of x in G. For a set X of vertices of G we put $N_G[X] := \bigcup_{x \in X} N_G[x]$ and $N_G(X) := N_G[X] - X$, we denote by $G[X]$ the subgraph of G induced by X, and we set $G - X := G[V(G) - X]$.

A *path* $P = \langle x_0, \ldots, x_n \rangle$ is a graph with $V(P) = \{x_0, \ldots, x_n\}$, $x_i \neq x_j$ if $i \neq j$, and $E(P) = \{x_i x_{i+1} : 0 \leq i < n\}$. A path $P = \langle x_0, \ldots, x_n \rangle$ is called an (x_0, x_n)*-path*, x_0 and x_n are its *endvertices*, while the other vertices are called its *inner* vertices, $n = |E(P)|$ is the *length* of P.

A *cycle* C with $V(C) = \{x_1, \ldots, x_n\}$, $x_i \neq x_j$ if $i \neq j$, and $E(C) = \{x_i x_{i+1} : 1 \leq i < n\} \cup \{x_n x_1\}$, is denoted by $\langle x_1, \ldots, x_n, x_1 \rangle$. The non-negative integer $n = |E(C)|$ is the *length* of C, and a cycle of length n is called an n*-cycle* and is often denoted by C_n.

Let G be a connected graph. The usual *distance* between two vertices x and y, that is, the length of any (x, y)*-geodesic* (=shortest (x, y)-path) in G, is denoted by $d_G(x, y)$. A connected subgraph H of G is *isometric* in G if $d_H(x, y) = d_G(x, y)$ for all vertices x and y of H. The *(geodesic) interval* $I_G(x, y)$ between two vertices x and y of G is the set of vertices of all (x, y)-geodesics in G.

2.2. Convexities

A *convexity* on a set X is an algebraic closure system $\mathcal{C}$ on X. The elements of $\mathcal{C}$ are the *convex sets* and the pair $(X, \mathcal{C})$ is called a *convex structure*. See van de Vel [19] for a detailed study of abstract convex structures. Several kinds of graph convexities, that is convexities on the vertex set of a graph G, have already been investigated. We will principally work with the *geodesic convexity*, that is the convexity on $V(G)$ which is induced by the geodesic interval operator I_G. In this convexity, a subset C of $V(G)$ is convex provided it contains the geodesic interval $I_G(x, y)$ for all $x, y \in C$. The *convex hull* $co_{\mathcal{G}}(A)$ of a subset A of $V(G)$ is the smallest convex set which contains A. The convex hull of a finite set is called a *polytope*. A subset H of $V(G)$ is a *half-space* if H and $V(G) - H$ are convex. We denote by $\mathcal{I}_G$ the pre-hull operator of the geodesic convex structure of G, i.e. the self-map of $\mathcal{P}(V(G))$ such that $\mathcal{I}_G(A) := \bigcup_{x,y \in A} I_G(x, y)$ for each $A \subseteq V(G)$. The convex hull of a set $A \subseteq V(G)$ is then $co_G(A) = \bigcup_{n \in \mathbb{N}} \mathcal{I}_G^n(A)$. Furthermore we say that a subgraph of a graph G is *convex* if its vertex set is convex, and by the *convex hull* $co_G(H)$ of a subgraph H of G we mean the smallest convex subgraph of G containing H as a subgraph, that is

$$co_G(H) := G[co_G(V(H))].$$

2.3. Netlike partial cubes

First we recall some properties of *partial cubes*, that is of isometric subgraphs of hypercubes. Partial cubes are particular connected bipartite graphs.

For an edge ab of a graph G, let

$$W_{ab}^G := \{x \in V(G) : d_G(a, x) < d_G(b, x)\},$$
$$U_{ab}^G := N_G(W_{ba}^G).$$

If no confusion is likely, we will simply denote W_{ab}^G and U_{ab}^G by W_{ab} and U_{ab}, respectively. Note that the sets W_{ab} and W_{ba} are disjoint and that $V(G) = W_{ab} \cup W_{ab}$ if G is bipartite and connected.

Two edges xy and uv are in the Djoković–Winkler relation Θ if

$$d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u).$$

If G is bipartite, the edges xy and uv are in relation Θ if and only if $d_G(x, u) = d_G(y, v)$ and $d_G(x, v) = d_G(y, u)$. The relation Θ is clearly reflexive and symmetric.

Theorem 2.1 (*Djoković [6, Theorem 1] and Winkler [20]*). *A connected bipartite graph G is a partial cube if and only if it has one of the following properties:*

(i) *For every edge ab of G, the sets W_{ab} and W_{ba} are convex (and thus are half-spaces).*
(ii) *The relation Θ is transitive.*

We denote by $CV(G)$ (resp. $3V(G)$) the set of vertices of a graph G which belong to a cycle of G (resp. whose degree is at least 3). We say that a set $A \subseteq V(G)$ is $\mathcal{C}$-*convex* (resp. (3)-*convex*) if $CV(G[\mathcal{I}_G(A)]) \subseteq A$ (resp. $3V(G[\mathcal{I}_G(A)]) \subseteq A$). The set of $\mathcal{C}$-convex subsets of $V(G)$ and the one of (3)-convex subsets of $V(G)$ are convexities on $V(G)$ which are finer than the geodesic convexity.

Lemma 2.2 (*Polat [17, Corollary 2.7]*). *If A is a $\mathcal{C}$-convex set of a connected graph G, then $\mathcal{I}_G(A)$ is convex.*

Bandelt characterized median graphs as bipartite graphs G for which the sets U_{ab} and U_{ba} are convex for each edge ab of G. By relaxing the type of convexity in Bandelt's characterization of a median graph we obtain what we call a netlike partial cube.

Definition 2.3. We say that a partial cube G is *netlike* if U_{ab} and U_{ba} are $\mathcal{C}$-convex for each edge ab of G.

Thus median graphs are netlike partial cubes. Clearly even cycles are also netlike partial cubes, and moreover any convex subgraph of a netlike partial cube is a netlike partial cube. Among different characterizations of netlike partial cubes we will need the following one:

Proposition 2.4 (*Polat [16, Theorem 3.10]*). *A partial cube G is netlike if and only if it has the following two properties:*

(i) *For each edge ab of G, the sets U_{ab} and U_{ba} are* (3)-*convex.*
(ii) *The convex hull of each non-convex isometric cycle of G is a hypercube.*

An induced subgraph H (or its vertex set) of a graph G is said to be *gated* if, for each $x \in V(G)$, there exists a vertex y (the *gate* of x) in H such that $y \in I_G(x, z)$ for every $z \in V(H)$. Obviously, every gated subgraph is convex, but the converse, which holds if G is a median graph, is generally not true if G is a netlike partial cube. However, if G is a netlike partial cube, then any hypercube is clearly gated and, by [16, Corollary 6.4], any convex cycle of G is gated. Hence, in all the results of this paper, we could replace "convex cycle" by the seeming stronger expression "gated cycle". The following result characterizes the convex subgraphs of a netlike partial cube which are gated.

Proposition 2.5 (*Polat [16, Theorem 6.2]*). *A convex subgraph H of a netlike partial cube is gated if and only if every convex cycle which has at least three vertices in common with H is a cycle of H.*

Proposition 2.6 (*Bandelt [1, Proposition 2.4]*). *The gated subgraphs of a graph have the Helly property, that is, every finite family of gated subgraphs that pairwise intersect have a non-empty intersection.*

A netlike partial cube G such that, for each edge ab, $\mathcal{I}_G(U_{ab})$ and $\mathcal{I}_G(U_{ba})$ induce trees, is called a *linear partial cube*. Benzenoid graphs and cellular bipartite graphs are instances of linear partial cubes. A *benzenoid graph* is a particular connected induced subgraph of the hexagonal grid, viz., a connected plane graph in which all inner faces are regular hexagons, each vertex belongs to a hexagon and all inner vertices have degree 3. The *cellular bipartite graphs*, which were defined and studied by Bandelt and Chepoi [2], are the graphs which can be obtained from a collection of single edges and even cycles by successive gated amalgamations.

Condition (ii) of Proposition 2.4 leads us to the following characterizations of two particularly important instances of netlike partial cubes: on the one hand, *a median graph is a netlike partial cube whose only convex cycles are 4-cycles* [16, Corollary 7.2], and on the other hand, *a linear partial cube is a netlike partial cube whose isometric cycles are convex* [16, Theorem 7.4].

Finally note that, because a partial cube G is an isometric subgraph of some hypercube Q, if a triple (x, y, z) of vertices of G has a median m in G, then m is the median of (x, y, z) in Q, and thus is unique.

3. Regular netlike partial cubes

We first characterize the regular netlike partial cubes.

Lemma 3.1 (*Polat [16, Lemma 6.1]*). *Let ab be an edge of a netlike partial cube G. Then any convex cycle of $G[U_{ab}]$ is a 4-cycle.*

Lemma 3.2. *Let G be a finite netlike partial cube. Then there exists an edge ab of G such that $W_{ab} = \mathcal{I}_G(U_{ab})$.*

Proof. Let ab be an edge of G such that $|W_{ab}|$ is as small as possible. Suppose that $W_{ab} \neq \mathcal{I}_G(U_{ab})$. Let $x \in \mathcal{I}_G(U_{ab})$ and $y \in N_G(x) \cap (W_{ab} - \mathcal{I}_G(U_{ab}))$. Then the edge xy is not Θ-equivalent to any edge of $G[\mathcal{I}_G(U_{ab})]$, and thus to any edge of $G[W_{ba} \cup \mathcal{I}_G(U_{ab})]$, by [16, Lemma 4.6]. Then $W_{ba} \cup \mathcal{I}_G(U_{ab}) \subseteq W_{xy}$. Hence $W_{xy} \subset W_{ab}$, contrary to the fact that $|W_{ab}|$ is minimum. Therefore $W_{ab} = \mathcal{I}_G(U_{ab})$. $\square$

Theorem 3.3. *The finite regular netlike partial cubes are the hypercubes and the even cycles.*

Proof. Clearly hypercubes and even cycles are regular netlike partial cubes. Let G be a finite regular netlike partial cube which is not a cycle. Hence $\delta_G(x) \geq 3$ for every vertex x of G. We will prove by induction on $|V(G)|$ that G is a hypercube. This is obvious if $|V(G)| = 1$. Suppose that this is true if $|V(G)| \leq n$ for some positive integer n. Let G be such that $|V(G)| = n + 1$. By Lemma 3.2, there exists an edge ab of G such that $W_{ab} = \mathcal{I}_G(U_{ab})$.

It follows, since $\delta_G(x) \geq 3$ for every $x \in V(G)$, that $\mathcal{I}_G(U_{ab}) = U_{ab}$ because G is netlike. Hence $W_{ab} = U_{ab}$, and thus, by the properties of partial cubes, $G = K_2 \Box G[U_{ab}]$. Because U_{ab} is convex, it follows that $G[U_{ab}]$ is netlike. Hence $G[U_{ab}]$ is a regular netlike partial cube different from a cycle of length greater than 4 by Lemma 3.1, and thus it is a hypercube by the induction hypothesis. Therefore G is a hypercube. $\square$

This is obviously not necessarily true for an infinite netlike partial cube. In fact, in addition to the infinite regular median graphs which are not hypercubes such as the infinite regular trees, there are also pure infinite regular netlike partial cubes: the hexagonal grid for example.

Corollary 3.4. *A finite regular netlike subgraph of a netlike partial cube is a convex cycle or a finite hypercube.*

This is a consequence of Theorem 3.3, of the fact that a netlike subgraph is an isometric subgraph, and of Proposition 2.4.

4. Fixed regular netlike subgraphs theorems in finite netlike partial cubes

Our approach and treatment of fixed subgraphs properties for finite netlike partial cubes will be similar to that used by Brešar et al. for tree-like partial cubes [4].

We first recall some definitions. If x and y are two vertices of a finite graph G, then x is said to be *dominated* by y in G if $N_G[x] \subseteq N_G[y]$. We say that a graph G is *dismantlable* if its vertices can be linearly ordered $x_0, \ldots, x_n$ so that, for each $i < n$, the vertex x_i is dominated by x_{i+1} in the subgraph of G induced by $\{x_i, \ldots, x_n\}$. The enumeration $x_0, \ldots, x_n$ is called a *dismantling enumeration* of the vertices of G.

Given a graph G, we denote by $G^\diamond$ the graph having the same vertex set as G and where two vertices are adjacent if and only if they belong to a common hypercube or a common convex cycle of G.

Proposition 4.1. *If G is a finite netlike partial cube, then $G^\diamond$ is dismantlable.*

Proof. The proof will be by induction on the order $|V(G)|$ of G. This is obvious if $|V(G)| = 1$. Suppose that this holds for any netlike partial cube whose order is at most n, for some positive integer n. Let G be a netlike partial cube such that $|V(G)| = n + 1$.

By Lemma 3.2, there exists an edge ab of G such that $W_{ab} = \mathcal{I}_G(U_{ab})$. If $W_{ab} \neq U_{ab}$, we first consider the elements of $W_{ab} - U_{ab}$. Let $x \in W_{ab} - U_{ab}$. Then $\delta_{G[W_{ab}]}(x) = 2(= \delta_G(x))$ because U_{ab} is (3)-convex since G is netlike (see Proposition 2.4). Hence there are two vertices $u, v \in U_{ab}$ and a (u, v)-geodesic P containing x such that every internal vertex of P has degree 2. Let u' and v' be the neighbors in U_{ba} of u and v, respectively, and let Q be a (u', v')-geodesic. Then $\langle u, u'\rangle \cup Q \cup \langle v', v\rangle \cup P$ is a convex cycle of G since every vertex of P has degree 2. Therefore x is dominated in $G^\diamond$ by any vertex of this cycle, and in particular by u' and v'.

Let $x_0, \ldots, x_i$ be an enumeration of the vertices of $W_{ab} - U_{ab}$. In the subgraph $G - \{x_0, \ldots, x_i\}$ (that is in G if $W_{ab} = U_{ab}$), each vertex u of U_{ab} is clearly dominated by its neighbor u' in U_{ba} because u' belongs to every hypercube (which is either a Q_1 or a Q_2) to which belongs u. Let $x_{i+1}, \ldots, x_j$ be an enumeration of the vertices in U_{ab}, and let $H := G - \{x_0, \ldots, x_j\}$.

This subgraph H is clearly netlike. Consequently, by the induction hypothesis, H is dismantlable. Let $x_{j+1}, \ldots, x_{n+1}$ be a dismantling enumeration of $V(H)$. Then $x_0, \ldots, x_{n+1}$ is a dismantling enumeration of the vertices of G. □

We say that a hypercube of a netlike partial cube G is *maximal* if it is not a proper subgraph of a hypercube or of a convex cycle of G. We denote by $\mathcal{H}(G)$ the graph whose vertex set is the set of all convex cycles and maximal hypercubes of G, and such that two vertices are adjacent if and only if they have a non-empty intersection. The graph $\mathcal{H}(G)$ is the clique graph of $G^\diamond$, that is the intersection graph of the maximal simplices (i.e. complete subgraphs) of $G^\diamond$. From Proposition 4.1 and Bandelt and Prisner's result [3, Proposition 2.6] stating that the clique graph of a dismantlable graph is again dismantlable, we obtain:

Corollary 4.2. *If G is a finite netlike partial cube, then $\mathcal{H}(G)$ is dismantlable.*

We recall that, if G and H are two graphs, a map $f : V(G) \to V(H)$ is a *contraction* (*weak homomorphism* in [7]) if f preserves or contracts the edges, i.e., if $f(x) = f(y)$ or $f(x)f(y) \in E(H)$ whenever $xy \in E(G)$. Notice that a contraction $f : G \to H$ is a non-expansive map between the metric spaces $(V(G), d_G)$ and $(V(H), d_H)$, i.e. $d_H(f(x), f(y)) \leq d_G(x, y)$ for all $x, y \in V(G)$. We say that a self-contraction f of G *fixes* a subgraph H of G if $f(H) = H$, and it *fixes* a subset A of $V(G)$ if it fixes the induced subgraph $G[A]$.

Theorem 4.3. *Any finite netlike partial cube G contains a convex cycle or a hypercube which is fixed by every automorphism of G.*

Proof. Each automorphism of G clearly induces an automorphism of $\mathcal{H}(G)$. By Corollary 4.2 and [9, Theorem 4.8], there exists a finite simplex $\mathcal{S}$ of $\mathcal{H}(G)$ which is fixed by every automorphism of this graph. By the definition of $\mathcal{H}(G)$, the elements of $\mathcal{S}$ are pairwise non-disjoint, and thus, by Proposition 2.6, they have a non-empty intersection H. By Proposition 2.5 and because the elements of $\mathcal{H}(G)$ are gated, this intersection H is a convex cycle or a hypercube. Moreover, this subgraph is clearly fixed by every automorphism of G. Note that, if H is a convex cycle, then it is the only element of $\mathcal{S}$. □

Theorem 4.4. *Every self-contraction of a finite netlike partial cube G fixes a convex cycle or a hypercube of G.*

Proof. Let f be a self-contraction of G. Because G is finite, $f^n(G) = f^{n+1}(G)$ for some non-negative integer n. Let $H := f^n(G)$. Then there exists $p \geq n$ such that $f^p(x) = x$ for every $x \in V(H)$, and thus f^p is a retraction of G onto H. By [17, Theorem 3.1], H is a netlike partial cube. Hence, by Theorem 4.3, H contains a convex cycle or a hypercube F which is fixed by every automorphism of H, and thus by f. We are done because a hypercube of H is obviously a hypercube of G, and a convex cycle of H is an isometric cycle of G whose convex hull – which is this cycle itself or a hypercube by Proposition 2.4 – is fixed by f. □

5. Geodesic topology

In order to extend the results of the preceding section to infinite netlike partial cubes, we introduce some topological concepts. Most of the results of this section are analogous to some of the results of [15], but almost all proofs are different.

A *ray* or *one-way infinite path* $\langle x_0, x_1, \ldots\rangle$ is a graph with $V(P) = \{x_n : n \in \mathbb{N}\}$, $x_i \neq x_j$ if $i \neq j$, and $E(P) = \{x_n x_{n+1} : n \in \mathbb{N}\}$.

An infinite subset S of $V(G)$ is *concentrated* in G if any two infinite subsets of S cannot be separated by removing finitely many vertices. For example the vertex set of any ray of a graph G is concentrated. Note that any infinite subset of a concentrated set is also concentrated.

A vertex x of a graph G *geodesically dominates* (*geo-dominates* for short) a subset A of $V(G)$ if, for every finite $S \subseteq V(G-x)$, there exists an $a \in (A-\{x\}) \cap V(\mathcal{C}_G(x))$ (where $\mathcal{C}_G(x)$ denotes the component of G containing the vertex x) such that $S \cap I_G(x, a) = \emptyset$.

Proposition 5.1 (*Polat [11, Theorem 3.9]*). *Let G be a graph. The following assertions are equivalent:*

(i) *G contains no isometric rays.*
(ii) *The vertex set of every ray of G is geo-dominated.*
(iii) *Every concentrated set of G is geo-dominated.*

We say that a set A of vertices of a graph G is *finitely geo-dominated* if the set of vertices which geo-dominate A is finite and non-empty. We also say that a ray is (finitely) geo-dominated if its vertex set is (finitely) geo-dominated.

We recall two facts that we will frequently use from now on:

- any partial cube is interval-finite, that is each of its interval is finite;
- each polytope of a partial cube is finite.

The following result is obvious.

Lemma 5.2 (*Polat [13, Proposition 4.1]*). *Let G be an interval-finite graph. Then a vertex x of G geo-dominates a subset A of $V(G)$ if and only if there exists an infinite subset B of A such that $I_G(x, a) \cap I_G(x, b) = \{x\}$ for every pair $\{a, b\}$ of distinct elements of B.*

Lemma 5.3. *Let G be a netlike partial cube, and A an infinite subset of $V(G)$ which is geo-dominated by a vertex m. For each $w \in V(G)$ there exists no infinite subset B of A such that $I_G(m, a) \cap I_G(m, b) = \{m\}$ for every pair $\{a, b\}$ of distinct elements of $B \cup \{w\}$, and with $m \notin I_G(a, w)$ for every $a \in B$.*

Proof. Let $w \in V(G)$. Assume that there exists an infinite $B \subseteq A$ such that $I_G(m, a) \cap I_G(m, b) = \{m\}$ for every pair $\{a, b\}$ of distinct elements of $B \cup \{w\}$, and with $m \notin I_G(a, w)$ for every $a \in B$.

(a) Let uv be an edge of G such that $w \in W_{uv}$ and $m \in W_{vu}$. Suppose that there are at least two elements of B in W_{uv}. Let K be a copoint at m containing W_{uv}. Then $K = W_{xy}$ for some edge xy of G, and $m \in \mathcal{I}_G(U_{yx})$. Because $|B \cap W_{xy}| \geq 2$, and since $I_G(m, b) \cap I_G(m, c) = \{m\}$ for every pair $\{b, c\}$ of distinct elements of $B \cup \{w\}$, it follows that the degree of m in $G[\mathcal{I}_G(U_{yx})]$ is at least 3. Hence $m \in U_{yx}$ since G is netlike. Let m' be the neighbor of m in U_{xy}. Then $m' \in I_G(m, b)$ for every $b \in B \cap W_{xy}$, contrary to the property of m. Therefore $|B \cap W_{uv}| \leq 1$.

(b) Let $\langle w_0, \ldots, w_n\rangle$ be a (w, m)-geodesic with $w_0 = w$ and $w_n = m$. We will prove by induction that $w_i \in I_G(w, b)$ for all $b \in B'$ for some subset B' of B such that $|B - B'| \leq 2$. This is obvious for $i = 0$. Let i be such that $0 \leq i < n$. Suppose that this is true for every $j \leq i$. Clearly $w_0 \in W_{w_i w_{i+1}}$ and $m \in W_{w_{i+1} w_i}$. By (a), we have $|B \cap W_{w_i w_{i+1}}| \leq 1$. Because w_{i+1} is the neighbor of w_i in $U_{w_{i+1} w_i}$, it follows that $w_{i+1} \in I_G(w_i, b)$ for every $b \in B \cap W_{w_{i+1} w_i} =: B'$. Hence $w_{i+1} \in I_G(w, b)$ for every $b \in B'$ by the induction hypothesis. Finally $m = w_n \in I_G(w, b)$ for every $b \in B'$, contrary to the properties of the set B. □

For a subset A of vertices of a graph G, we denote by $M_G(A)$ the set of all vertices belonging to $I_G(a, b)$ for every pair $\{a, b\}$ of distinct elements of A. Note that, if G is a partial cube and if $|A| \geq 3$, then $|M_G(A)| \leq 1$ because, if $m \in M_G(A)$, then m is the median of any triple of elements of A, and thus is unique.

Lemma 5.4. *Let G be a netlike partial cube, and let A be an infinite subset of $V(G)$. Then a vertex m of G geo-dominates A if and only if $m \in M_G(B)$ for some infinite subset B of A.*

Proof. Suppose that m geo-dominates A. Without loss of generality we can assume, by Lemma 5.2 since G is interval-finite, that $I_G(m, a) \cap I_G(m, b) = \{m\}$ for every pair $\{a, b\}$ of distinct elements of A. Suppose that $m \notin M_G(B)$ for every infinite subset B of A. Then, for every infinite $B \subseteq A$, there exist $a, b \in B$ such that $a \neq b$ and $m \notin I_G(a, b)$. Hence, by Ramsey's theorem, A contains an infinite subset B such that $m \notin I_G(a, b)$ for every pair $\{a, b\}$ of distinct elements of B. But this is in contradiction with Lemma 5.3 where w is any element of B.

Consequently $m \in M_G(B)$ for some infinite $B \subseteq A$. The converse is due to the fact that, if $m \in I_G(a, b)$, then obviously $I_G(m, a) \cap I_G(m, b) = \{m\}$. □

In [11] we endowed the vertex set of a graph with a topology, called the *geodesic topology*, where a subset A of $V(G)$ is closed if and only if every vertex which geo-dominates A belongs to A. In particular we proved that *the geodesic space $V(G)$ is compact if and only if G contains no isometric rays*.

Theorem 5.5. *Let G be a netlike partial cube. Then the geodesic topology on $V(G)$ is the topology (in terms of closed sets) generated by the convex subsets of $V(G)$ as a subbase.*

Proof. We have to show that every geodesically closed set (i.e., closed with respect to the geodesic topology) is an intersection of a finite union of convex sets. By Lemma 5.4 every convex subset of $V(G)$ is geodesically closed. Let A be an infinite subset of $V(G)$ and u a vertex of $G - A$ which belongs to the geodesic closure of A. Hence u geo-dominates A and thus, by Lemma 5.4, $u \in M_G(B)$ for some infinite $B \subseteq A$. Let $(C_i)_{1 \leq i \leq n}$ be a finite family of convex sets whose union contains A. Since B is infinite, there are two elements b and b' of B which belongs to some C_i. Hence $I_G(b, b') \subseteq C_i$ by the convexity of C_i, and thus $u \in C_i$. Therefore u belongs to the intersection of every finite union of convex sets which contains A. □

This geodesic topology corresponds, for median graphs, to the topology introduced by Tardif in [18]. We will omit the proofs of the following two lemmas because, except for the references, they are analogous to those of Lemmas 4.5 and 4.6 of [15], respectively.

Lemma 5.6. *Let G be a netlike partial cube containing no isometric rays. Then, for every infinite subset A of vertices of G, there exists a vertex $x \in A$ which does not geo-dominate $A - \{x\}$. In particular, for a netlike partial cube G, the compactness of the geodesic space $V(G)$ implies that this space is scattered.*

Lemma 5.7. *Let G be a netlike partial cube containing no isometric rays. Then every concentrated subset of $V(G)$ is finitely geo-dominated.*

We obtain immediately:

Corollary 5.8. *A netlike partial cube G contains no isometric rays if and only if every ray of G is finitely geo-dominated.*

6. Fixed finite regular netlike subgraphs theorems in infinite netlike partial cubes

Proposition 6.1 (*Polat [15, Proposition 3.1]*). *If every self-contraction of a connected graph G stabilizes a non-empty finite set of vertices of G, then this graph contains no isometric rays and no infinite simplices.*

Proposition 6.2 (*Polat [15, Theorem 3.6]*). *Every self-contraction of a connected graph G whose rays are all finitely geo-dominated strictly stabilizes a non-empty finite set of vertices of G.*

Proposition 6.3 (*Polat [13, Theorem 3.3]*). *Let G be a connected graph such that the geodesic space $V(G)$ is compact and scattered. Then there exists a non-empty finite set of vertices of G that is fixed by every automorphism of G.*

Lemma 6.4. *Let G be a netlike partial cube. If there exists a non-empty finite set of vertices which is fixed by every automorphism of G, then there exists a convex cycle or a finite hypercube which is fixed by every automorphism of G.*

Proof. Let S be a non-empty finite set of vertices which is fixed by every automorphism of G. Then the polytope $co_G(S)$, which is finite since G is a partial cube, is clearly fixed by all automorphisms of G. Therefore $H := G[co_G(S)]$ is a finite convex subgraph of G, and thus a netlike subgraph, which is fixed by all automorphisms of G.

Hence, by Theorem 4.3, H contains a convex cycle or a hypercube F which is fixed by every automorphism of H, and thus of G. Note that a cycle which is convex in H is also convex in G because H is convex. □

Theorem 6.5. *Let G be a netlike partial cube containing no isometric rays. Then there exists a convex cycle or a finite hypercube which is fixed by every automorphism of G.*

Proof. Because G contains no isometric rays, it follows that the geodesic space $V(G)$ is compact, and moreover that it is scattered by Lemma 5.6. Therefore it follows, by Proposition 6.3 that there exists a non-empty finite set which is fixed by all automorphisms of G. The result is then a consequence of Lemma 6.4. □

Note that, on account of Lemma 6.4, other fixed finite regular netlike subgraph theorems quite analogous to [15, Theorem 5.13, Proposition 5.14 and Corollary 5.15] for weakly median graphs could be stated and easily proved, so we leave them to the reader.

Theorem 6.6. *Every self-contraction of a netlike partial cube G fixes a convex cycle or a finite hypercube of G if and only if G contains no isometric rays.*

Proof. By Proposition 6.1 we have only to prove the sufficiency. Let f be a self-contraction of G. By Corollary 5.8 and Proposition 6.2 there exists a non-empty finite set S of vertices of G which is fixed by f. The subgraph $H := G[co_G(S)]$ is a finite netlike subgraph of G. Furthermore the restriction f' of f to $V(H)$ is a self-contraction of H. Therefore, by Theorem 4.4, there exists a convex cycle or a hypercube of H, and thus of G, which is fixed by f', hence by f. □

In next results we will use the following notations: for a self-contraction f of a graph G, $x \in V(G)$ and $A \subseteq V(G)$ we set

$$[x]_f := \{f^n(x) : n \in \mathbb{N}\}$$

and

$$A_f := \{x \in A : [x]_f \subseteq A \text{ and } f^n(x) = x \text{ for some } n > 0\}.$$

Lemma 6.7. *Let G be a netlike partial cube, A a non-empty subset of $V(G)$ such that $G[A]$ is an isometric subgraph of G, and f a self-contraction of G that fixes A. Then:*

(i) *$G[A_f]$ is an isometric subgraph of G.*
(ii) *$G[A_f]$ is a netlike partial cube.*
(iii) *A_f is geodesically closed.*

Proof. (i) is Lemma 3.10(i) of [12].

(ii) Let ab be an edge of $H := G[A_f]$. We have to prove that U^H_{ab} (and U^H_{ba}) are $\mathcal{C}$-convex. Suppose that there exists a vertex c which belongs to a cycle C of $H[\mathcal{I}_H(U^H_{ab})]$. Then $c \in I_H(u, v)$ for some vertices $u, v \in U^H_{ab}$. Let u' and v' be the neighbors of u and v in U^H_{ba}, respectively. Clearly $u, v \in U^G_{ab}$ and $u', v' \in U^G_{ba}$. Moreover, since H is an isometric subgraph of G by (i), it follows that $c \in I_G(u, v)$ and C is a cycle of $G[\mathcal{I}_G(U^G_{ab})]$. Therefore $c \in U^G_{ab}$ because G is netlike. Let d be the neighbor of c in U^G_{ba}. Then $\{d\} = N_G(c) \cap I_G(u', v')$. By the definition of A_f, there exists $n > 0$ such that $f^n(c) = c$, $f^n(u') = u'$ and $f^n(v') = v'$. Then $f^n(d) \in N_H[c] \cap I_H(u', v') \subseteq N_G[c] \cap I_G(u', v')$. Hence, $f^n(d) = d$ since $c \notin I_G(u', v')$. Therefore $c \in U^H_{ab}$. It follows that U^H_{ab} is $\mathcal{C}$- convex. Consequently $G[A_f]$ is a netlike partial cube.

(iii) Let m be a vertex of G that geo-dominates A_f. Then, by Lemma 5.4, $m \in M_G(B)$ for some infinite subset B of A_f. Without loss of generality we can suppose that B is countably infinite. Let $S_0 \subset S_1 \subset \dots$ be an infinite sequence of finite subsets of B such that $\bigcup_{i\geq 0} S_i = B$. For every $i \geq 0$, since S_i is a finite subset of A_f, and because G is interval-finite, there exists a positive integer n_i such that $f^{n_i}(m) \in M_{G[A_f]}(S_i)$. Clearly $M_{G[A_f]}(S_j) \subseteq M_{G[A_f]}(S_i)$ if $i \leq j$ since $S_i \subseteq S_j$. Therefore by the finiteness of each set $M_{G[A_f]}(S_i)$, there exists a positive integer N such that $f^N(m) \in M_{G[A_f]}(S_i)$ for every $i \geq 0$. This implies that $f^N(m) \in M_{G[A_f]}(B)$. By (i), $G[A_f]$ is an isometric subgraph of G, then $M_{G[A_f]}(B) \subseteq M_G(B)$. Hence $f^N(m) = m$ because $M_G(B)$ has exactly one element since G is a partial cube. Therefore $m \in M_{G[A_f]}(B)$, which proves that A_f is a closed set. □

Theorem 6.8. *Let $\mathcal{F}$ be a commuting family of self-contractions of a netlike partial cube G containing no isometric rays. Then there exists a convex cycle or a finite hypercube which is fixed by every element of $\mathcal{F}$.*

Proof. For every $f \in \mathcal{F}$, the set V_f, where V stands for $V(G)$, is non-empty by Theorem 6.6, and such that $G[V_f]$ is an isometric netlike subgraph of G by Lemma 6.7. Therefore $G[V_f]$ is a netlike partial cube containing no isometric rays. If $g \in \mathcal{F}$ commutes with f on V_f, and if $x \in V_f$, then $f^p(g(x)) = g(f^p(x)) = g(x)$ for any $p \geq 0$ such that $f^p(x) = x$. Thus $g(V_f) \subseteq V_f$. Hence, since $G[V_f]$ is a netlike partial cube that contains no isometric rays, then, by Theorem 6.6, it follows that g fixes a non-empty finite subset of V_f. Therefore $V_f \cap V_g = (V_f)_g$ $(= (V_g)_f)$ is non-empty and $G[V_f \cap V_g]$ is an isometric netlike subgraph of G by Lemma 6.7. Note that $[x]_f \cup [x]_g \subseteq V_f \cap V_g$ for every $x \in V_f \cap V_g$. Hence the restrictions of f and of g to $V_f \cap V_g$ are automorphisms of $G[V_f \cap V_g]$. Inductively, for any non-empty finite $\mathcal{H} := f_1, \dots, f_n \subseteq \mathcal{F}$, the set $V_{\mathcal{H}} := \bigcap_{f\in\mathcal{H}} V_f = (\dots(V_{f_1})\dots)_{f_n}$ is non-empty and such that $G[V_{\mathcal{H}}]$ is an isometric netlike subgraph of G. Therefore $V_{\mathcal{F}} := \bigcap_{f\in\mathcal{F}} V_f \neq \emptyset$ since the geodesic space $V(G)$ is compact and, by Lemma 6.7(iii), the sets V_f's are geodesically closed. Furthermore, the restriction of every $f \in \mathcal{F}$ to $V_{\mathcal{F}}$ is an automorphism of $H := G[V_{\mathcal{F}}]$. Besides, since each $G[V_f]$ is an isometric netlike subgraph of G, and hence is interval-finite, we conclude that H, being the intersection of all $G[V_f]$'s, is also an isometric netlike subgraph of G which contains no isometric rays. Then, by Theorem 6.5, H contains a convex cycle or a finite hypercube which is fixed by every element of $\mathcal{F}$. We are done because a hypercube of H is obviously a hypercube of G, and a convex cycle of H is an isometric cycle of G whose convex hull – which is this cycle itself or a hypercube by Proposition 2.4 – is fixed by every element of $\mathcal{F}$. □

We complete this section by stating the particular forms of the three preceding theorems related to the two special cases of netlike partial cubes: the linear partial cubes and the median graphs. For the latest one these results were already obtained by Tardif [18]. In order to simplify these statements, the simplices K_1 and K_2, that is the 0-cube and the 1-cube, will be seen as the 0-cycle and the 2-cycle, more precisely if $V(K_1) = \{0\}$ and $V(K_2) = \{0, 1\}$, then $K_1 = C_0 = \langle 0\rangle$ and $K_2 = C_2 = \langle 0, 1, 0\rangle$.

Corollary 6.9. *Let G be a linear partial cube (resp. a median graph). We have the following properties:*

(i) *If G contains no isometric rays, then there exists a convex cycle (resp. a finite hypercube) which is fixed by every automorphism of G.*
(ii) *Every self-contraction of G fixes a convex cycle (resp. a finite hypercube) of G if and only if G contains no isometric rays.*
(iii) *Let $\mathcal{F}$ be a commuting family of self-contractions of a G. If G contains no isometric rays, then there exists a convex cycle (resp. a finite hypercube) which is fixed by every element of $\mathcal{F}$.*

7. Translating self-contractions

By Theorem 6.6, if a netlike partial cube G contains an isometric ray, then some self-contraction of G does not fix any non-empty finite set of vertices of G. Such a self-contraction which fixes no non-empty finite set of vertices is said to be *translating*, and a translating automorphism is called a *translation*. To study these particular self-contractions we need the concept of ends. The *ends* of a graph G are the classes of the equivalence relation defined on the set of all rays of G as follows: two rays R and R' are said to be *end-equivalent* if and only if there is a ray R'' whose intersections with R and R' are infinite, or equivalently if and only if $V(R)$ and $V(R')$ are infinitely linked in G. For an end ε of G and a finite $S \subseteq V(G)$ we denote by $\mathcal{C}_{G-S}(\varepsilon)$ the unique component of $G - S$ which contains an element of ε. By [10, Theorem 3.3], an infinite subset S of $V(G)$ is concentrated in G if there exists an end ε such that $S - V(\mathcal{C}_{G-F}(\varepsilon))$ is finite for every finite $F \subseteq V(G)$ (S is said to be *concentrated in* ε).

In [14] it was proved that a particular end is linked to each translating self-contraction of a graph in the following sense.

Lemma 7.1 (*Polat [14, Lemma 2.2]*). *If f is a translating self-contraction of a graph G, then there exists a unique end of G, called the* direction *of f and denoted by $\delta(f)$, such that, for every $x \in V(G)$, the set $\{f^n(x) : n \in \mathbb{N}\}$ is concentrated in $\delta(f)$.*

In this section we will characterize the ends of a netlike partial cube which are directions of translating self-contractions of this graph. We need several preliminary results.

Lemma 7.2 (*Chastand and Polat [5, Lemma 6.2]*). *Let ε be an end of a graph G whose polytopes are finite, and let $X \subseteq V(G)$ be concentrated in ε. Then every ray of the subgraph $G[co_G(X)]$ belongs to ε.*

Lemma 7.3. *Let G be a netlike partial cube, and let ε be an end of G such that no isometric ray of G belongs to ε. Then every ray in ε is finitely geo-dominated.*

Proof. Let $R \in \varepsilon$. By Lemma 7.2 and since every polytope of a partial cube is finite. it follows that every ray of $G[co_G(V(R))]$ belongs to ε. Then $G[co_G(V(R))]$ contains no isometric rays. Therefore, by Corollary 5.8, $V(R)$ is finitely geo-dominated in $G[co_G(V(R))]$, and thus in G because $co_G(V(R))$ is geodesically closed by Theorem 5.5. □

Proposition 7.4 (*Chastand and Polat [5, Proposition 6.6]*). *Let G be a graph such that, for any end ε, if no isometric ray of G belongs to ε, then each ray in ε is finitely geo-dominated. Then an end ε of G is the direction of a translating self-contraction of G if and only if there is an isometric ray which belongs to ε.*

The main result of this section follows immediately from Lemma 7.3 and Proposition 7.4.

Theorem 7.5. *Let G be a netlike partial cube. Then an end ε of G is the direction of a translating self-contraction of G if and only if there is an isometric ray which belongs to ε.*

Note that this result is in general not true for a partial cube which is not netlike. Consider for example the subdivision of an infinite simplex. The *subdivision graph* $S(G)$ of a graph G is the graph obtained from G by subdividing each edge of G by a single vertex. For any finite of infinite cardinal α, the subdivision graph $S(K_\alpha)$ is a partial cube. Indeed, each polytope is contained in a $S(K_n)$ for some non-negative integer n, $S(K_n)$ is finite and moreover it is a partial cube by [8, Proposition 2.1]. Furthermore it is not difficult to prove that $S(K_\alpha)$ is not netlike if $\alpha \geq 4$. Now take the partial cube $S(K_{\aleph_0})$. It is not netlike, has only one end, and contains no isometric rays because each of its rays is geo-dominated by any vertex of infinite degree. Let $V(K_{\aleph_0}) = \{x_n : n \in \mathbf{N}\}$ and, for $n \neq p$, let x_{np} be the vertex which subdivides the edge $x_n x_p$ of $K_{\aleph_0}$. Then the map f such that $f(x_n) = x_{n+1}$ and $f(x_{np}) = x_{n+1,p+1}$ is clearly a translating homomorphism of $S(K_{\aleph_0})$ whose direction is the unique end of this graph.

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